Journal of Computational and Applied Mathematics 233 (2009) 1046–1055

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence of numerical solutions to stochastic age-dependent population equations with Markovian switching

Ronghua Li [a,b,*], Ping-kei Leung [b], Wan-kai Pang [b]

[a] Department of Statistics, China University of Petroleum, Dongying 257061, Shandong, PR China

[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

ARTICLE INFO

Article history:
Received 9 June 2009
Received in revised form 26 August 2009

MSC:
60H15
37N25
65C20

Keywords:
Stochastic age-dependent population equations
Markovian switching
Numerical solutions
Convergence

ABSTRACT

In this paper, a class of stochastic age-dependent population equations with Markovian switching is considered. The main aim of this paper is to investigate the convergence of the numerical approximation of stochastic age-dependent population equations with Markovian switching. It is proved that the numerical approximation solutions converge to the analytic solutions of the equations under the given conditions. An example is given for illustration.

1. Introduction

Stochastic modelling has been widely used to model the phenomena arising in many branches of science and industry such as biology, economics, mechanics, electronics and telecommunications [1–3]. Recently, stochastic modelling with Markovian switching has received a great deal of attention [4–6]. Most types of stochastic modelling with Markovian switching are nonlinear and cannot have explicit solutions, so the construction of efficient computational methods is of great importance. For example, Yuan and Mao [7] obtained the convergence of the Euler–Maruyama method for stochastic differential equations with Markovian switching, Li et al. [8,9] discussed the convergence of the numerical solution to a stochastic delay differential equation with Poisson jump and Markovian switching, Zhou and Wu [10] investigated the convergence of numerical solutions to neutral stochastic delay differential equations with Markovian switching under the local Lipschitz condition. The equations they considered are stochastic (delay) differential equations with Markovian switching.

However, to the best of our knowledge, there are no numerical methods available for stochastic partial differential equations with Markovian switching. In this paper, we shall consider the convergence of numerical solutions for a class of stochastic age-dependent population equations with Markovian switching, although there are several articles dealing with the approximate schemes for stochastic age-dependent population equations [11–14].

In this paper, we shall extend the idea from the papers [10,15], to the numerical solutions for stochastic age-dependent population equations with Markovian switching. The main purpose of this paper is to investigate the convergence of the

* Corresponding author at: Department of Statistics, China University of Petroleum, Dongying 257061, Shandong, PR China.
E-mail address: lirongh64@yahoo.com.cn (R. Li).

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doi:10.1016/j.cam.2009.08.113

numerical approximation of stochastic age-dependent population equations with Markovian switching under the given conditions.

In Section 2, we shall collect some basic preliminaries which are essential for our analysis, and introduce the model of stochastic age-dependent population equations with Markovian switching. In Section 3, we give several lemmas which are useful for our main results. In Section 4, we shall show the main results that the numerical solutions will converge to the true solutions to stochastic age-dependent population equations with Markovian switching under the given conditions. In Section 5, one example is provided to illustrate our theory. Conclusion is given in Section 6.

2. Model description and preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state $S = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion W_t. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R_+.

Let

$$V = H^1([0, A]) \equiv \left\{ \varphi | \varphi \in L^2([0, A]), \frac{\partial \varphi}{\partial x_i} \in L^2([0, A]), \text{ where } \frac{\partial \varphi}{\partial x_i} \text{is generalized partial derivatives} \right\}.$$

V is a Sobolev space. $H = L^2([0, A])$ such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

V' is the dual space of V. We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V' respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V, V', and by $(\cdot,\cdot)$ the scalar product in H. Let W_t be a Wiener process defined on $(\Omega, \mathcal{F}, \mathcal{P})$ and taking its values in the separable Hilbert space K, with increment covariance W. For an operator $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be the space of all bounded linear operators from K into H, we denote by $\|B\|_2$ the Hilbert–Schmidt norm, i.e.

$$\|B\|_2^2 = \text{trace}(BWB^T).$$

Let $C = C([0, T]; H)$ be the space of all continuous function from $[0, T]$ into H with sup-norm $\|\psi\|_C = \sup_{0\leq s\leq T} |\psi|(s)$, $L_V^p = L^p([0, T]; V)$ and $L_H^p = L^p([0, T]; H)$.

Consider the following stochastic age-dependent population equations with Markovian switching

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{\partial P}{\partial a} = -\mu(t, a)P + f(r(t), P) + g(r(t), P)\dfrac{\mathrm{d}W_t}{\mathrm{d}t}, & (t, a) \in Q \\ P(0, a) = P_0(a), r(0) = i_0, & a \in [0, A], \\ P(t, 0) = \displaystyle\int_0^A \beta(t, a)P(t, a)\mathrm{d}a, & t \in [0, T], \end{cases} \tag{1}$$

or

$$\begin{cases} \mathrm{d}_t P = \left[-\dfrac{\partial P}{\partial a} - \mu(t, a)P + f(r(t), P) \right] \mathrm{d}t + g(r(t), P)\mathrm{d}W_t, & (t, a) \in Q, \\ P(0, a) = P_0(a), r(0) = i_0, & a \in [0, A], \\ P(t, 0) = \displaystyle\int_0^A \beta(t, a)P(t, a)\mathrm{d}a, & t \in [0, T], \end{cases} \tag{2}$$

where $T > 0$, $A > 0$, $Q = (0, T) \times (0, A)$, $\mathrm{d}_t P = \frac{\partial P}{\partial t}$. $P = P(t, a)$ denotes the population density of age a at time t, $\beta(t, a)$ denotes the fertility rate of females of age a at time t, $\mu(t, a)$ denotes the mortality rate of age a at time t and state $r(t)$. $f(r(t), P)$ denotes effects of external environment for population system. $g(r(t), P)$ is a diffusion coefficient. $f(i, \cdot) : S \times L_H^2 \to H$ be a family of nonlinear operators, $\mathcal{F}_t$-measurable almost surely in t. $g(i, \cdot) : S \times L_H^2 \to \mathcal{L}(K, H)$ is the family of nonlinear operator, $\mathcal{F}_t$-measurable almost surely in t. $P_0 \in L_H^2$.

The integral version of Eq. (2) is given by the equation

$$P_t = P_0 - \int_0^t \frac{\partial P_s}{\partial a}\mathrm{d}s - \int_0^t \mu(s,a)P_s\mathrm{d}s + \int_0^t f(r(s),P_s)\mathrm{d}s + \int_0^t g(r(s),P_s)\mathrm{d}W_s, \tag{3}$$

here $P_t = P(t,a)$.

For system (2) the discrete approximate solution on $t = 0, h, 2h, \ldots, Mh$ is defined by the iterative scheme

$$Q_t^{k+1} = Q_t^k - \frac{\partial Q_t^{k+1}}{\partial a}h - \mu(t_k,a)hQ_t^k + f(r_k^h, Q_t^k)h + g(r_k^h, Q_t^k)\Delta W_k, \tag{4}$$

with initial value $Q_t^0 = P(0,a)$, $Q^k(t,0) = \int_0^A \beta(t,a)Q_t^k\mathrm{d}a$, $r_k^h = r(kh)$, $k \geq 1$. Here, Q_t^k is the approximation to $P(t_k,a)$, for $t_k = kh$, the time increment is $h = \frac{T}{M} \ll 1$, Brownian motion increment is $\Delta W_k = W(t_{k+1}) - W(t_k)$.

For convenience, we shall extend the discrete numerical solutions to continuous time. We first define two-step functions

$$Z_t = Z(t,a) = \sum_{k=0}^{M-1} Q_t^k \mathbf{1}_{[kh,(k+1)h)}, \qquad \bar{r}(t) = \sum_{k=0}^{M-1} r_k^h \mathbf{1}_{[kh,(k+1)h)}(t), \tag{5}$$

where $\mathbf{1}_G$ is the indicator function for the set G. Then we define

$$Q_t = P_0 - \int_0^t \frac{\partial Q_s}{\partial a}\mathrm{d}s - \int_0^t \mu(s,a)Z_s\mathrm{d}s + \int_0^t f(\bar{r}(s),Z_s)\mathrm{d}s + \int_0^t g(\bar{r}(s),Z_s)\mathrm{d}W_s, \tag{6}$$

with $Q_0 = P(0,a)$, $Q(t,0) = \int_0^A \beta(t,a)Q_t\mathrm{d}a$, $Q_t = Q(t,a)$, $\bar{r}(0) = i_0$. It is straightforward to check that $Z(t_k,a) = Q_t^k = Q(t_k,a)$.

As the standing hypotheses we always assume that the following conditions are satisfied:

(i) $f(i,0) = 0, g(i,0) = 0, i \in S$;

(ii) (Lipschitz condition) there exists positive constants K_i such that $x, y \in C$, $i \in S$,

$$|f(i,y) - f(i,x)| \vee \|g(i,y) - g(i,x)\|_2 \leq K_i\|y - x\|_C, \quad a.e.t;$$

(iii) $\mu(t,a)$, $\beta(t,a)$ are continuous in $\bar{Q} = Q + \partial Q$ such that

$$0 \leq \mu_0 \leq \mu(t,a) \leq \bar{\alpha} < \infty, \qquad 0 \leq \beta(t,a) \leq \bar{\beta} < \infty.$$

In an analogous way to the corresponding proof presented in [16], we may establish the following existence and uniqueness conclusion: under the conditions (i)–(iii), Eq. (2) has a unique continuous solution $P(t,a)$ on $(t,a) \in Q$.

3. Several lemmas

In this section, we shall give several lemmas which are useful for the following main results.

As for $r(t)$, the following lemma is satisfied (see [17]).

Lemma 1. *Given $h > 0$, then $\{r_n^h = r(nh), n = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix.*

$$P(h) = (P_{ij}(h))_{N\times N} = \mathrm{e}^{h\Gamma}.$$

The following theorem gives the kth moment boundedness of solutions $P(t,a)$ to Eq. (2).

Lemma 2. *Under the conditions* (ii)–(iii), *there are constants $k \geq 2$ and $C_1 > 0$ such that*

$$E\left[\sup_{0\leq t\leq T} |P_t|^k\right] \leq C_1.$$

Proof. Applying Itô formula to $|P_t|^k$ yields

$$\begin{aligned}
|P_t|^k = {} & |P_0|^k + \int_0^t k|P_s|^{k-2}\left\langle -\frac{\partial P_s}{\partial a} - \mu(s,a)P_s, P_s\right\rangle \mathrm{d}s \\
& + \int_0^t k|P_s|^{k-2}(f(r(s),P_s),P_s)\mathrm{d}s + \int_0^t k|P_s|^{k-2}(P_s, g(r(s),P_s)\mathrm{d}W_s) \\
& + \int_0^t \frac{k(k-2)}{2}|P_s|^{k-4}\|(P_s, g(r(s),P_s)\mathrm{d}W_s)\|_2^2 + \int_0^t \frac{k}{2}|P_s|^{k-2}\|g(r(s),P_s)\|_2^2\mathrm{d}s
\end{aligned}$$

$$\leq |P_0|^k - k\int_0^t |P_s|^{k-2}\left\langle \frac{\partial P_s}{\partial a}, P_s\right\rangle \mathrm{d}s - k\mu_0\int_0^t |P_s|^{k-2}(P_s, P_s)\mathrm{d}s$$
$$+ k\int_0^t |P_s|^{k-2}(f(r(s), P_s), P_s)\mathrm{d}s + \frac{k}{2}\int_0^t |P_s|^{k-2}\|g(r(s), P_s)\|_2^2\mathrm{d}s$$
$$+ \frac{k(k-2)}{2}\int_0^t |P_s|^{k-4}\|(P_s, g(r(s), P_s)\mathrm{d}W_s)\|_2^2 + k\int_0^t |P_s|^{k-2}(P_s, g(r(s), P_s)\mathrm{d}W_s).$$

Note

$$-\left\langle \frac{\partial P_s}{\partial a}, P_s\right\rangle = -\int_0^A P_s\mathrm{d}_a(P_s) = \frac{1}{2}\left(\int_0^A \beta(s,a)P_s\mathrm{d}a\right)^2$$
$$\leq \frac{1}{2}\int_0^A \beta^2(s,a)\mathrm{d}a\int_0^A P_s^2\mathrm{d}a \leq \frac{1}{2}A^2\bar{\beta}^2|P_s|^2. \tag{7}$$

Therefore, by conditions (ii) and (iii), we get that

$$|P_t|^k \leq |P_0|^k + \frac{k}{2}(A^2\bar{\beta}^2 - 2\mu_0)\int_0^t |P_s|^k\mathrm{d}s + k\int_0^t |P_s|^{k-1}|f(r(s), P_s)|\mathrm{d}s$$
$$+ \frac{k(k-1)}{2}\int_0^t |P_s|^{k-2}\|g(r(s), P_s)\|_2^2\mathrm{d}s + k\int_0^t |P_s|^{k-2}(P_s, g(r(s), P_s)\mathrm{d}W_s)$$
$$\leq |P_0|^k + \frac{k}{2}(A^2\bar{\beta}^2 - 2\mu_0)\int_0^t |P_s|^k\mathrm{d}s + kK_i\int_0^t |P_s|^{k-1}\|P_s\|_C\mathrm{d}s$$
$$+ \frac{k(k-1)}{2}K_i\int_0^t |P_s|^{k-2}\|P_s\|_C^2\mathrm{d}s + k\int_0^t |P_s|^{k-2}(P_s, g(r(s), P_s)\mathrm{d}W_s).$$

Now, it follows that for any $t \in [0, T]$

$$E\sup_{0\leq s\leq t}|P_s|^k \leq E|P_0|^k + \frac{k}{2}(A^2\bar{\beta}^2 - 2\mu_0 + kK_i + K_i)\int_0^t E\sup_{0\leq s\leq t}|P_s|^k\mathrm{d}s + kE\sup_{0\leq s\leq t}\int_0^s |P_u|^{k-2}(P_u, g(r(u), P_u)\mathrm{d}W_u). \tag{8}$$

By the Burkholder–Davis–Gundy inequality, we have

$$E\left[\sup_{0\leq s\leq t}\int_0^s |P_u|^{k-2}(P_u, g(r(u), P_u)\mathrm{d}W_u)\right] = E\left[\sup_{0\leq s\leq t}\int_0^s (|P_u|^{\frac{k-2}{2}}P_u, |P_u|^{\frac{k-2}{2}}g(r(u), P_u)\mathrm{d}W_u)\right]$$
$$\leq 3E\left[\sup_{0\leq s\leq t}|P_s|^{k/2}\left(\int_0^t |P_s|^{k-2}\|g(r(s), P_s)\|_2^2\mathrm{d}s\right)^{1/2}\right]$$
$$\leq \frac{1}{2k}E\left[\sup_{0\leq s\leq t}|P_s|^k\right] + K_0\int_0^t |P_s|^{k-2}\|g(r(s), P_s)\|_2^2\mathrm{d}s$$
$$\leq \frac{1}{2k}E\left[\sup_{0\leq s\leq t}|P_s|^k\right] + K_0\cdot K_i^2\int_0^t E\sup_{0\leq s\leq t}|P_s|^{k-2}\|P_s\|_C^2\mathrm{d}s, \tag{9}$$

for some positive constant $K_0 > 0$. Thus, it follows from (8) and (9)

$$E\sup_{0\leq s\leq t}|P_s|^k \leq k(A^2\bar{\beta}^2 - 2\mu_0 + kK_i + K_i + 2K_0K_i^2)\int_0^t E\sup_{0\leq r\leq s}|P_r|^k\mathrm{d}s + 2E|P_0|^k, \quad \forall t \in [0, T].$$

Now, Gronwall's lemma obviously implies the required result with

$$C_1 = 2\mathrm{e}^{k(A^2\bar{\beta}^2 - 2\mu_0 + kK_i + K_i + 2K_0K_i^2)T}E|P_0|^k. \quad \Box$$

Lemma 3. *Under the conditions* (ii)–(iii), *there exists an constant $C_2 > 0$ such that*

$$E\sup_{t\in[0,T]}|Q_{t\wedge\upsilon_n}|^2 \leq C_2,$$

where $\tau_n = \inf\{t \geq 0 : |P_t| \geq n\}$, $\sigma_n = \inf\{t \geq 0 : |Q_t| \geq n\}$, $\upsilon_n = \tau_n \wedge \sigma_n$.

Proof. From (6), one can obtain

$$dQ_t = -\frac{\partial Q_t}{\partial a}dt - \mu(t,a)Z_t dt + f(\bar{r}(t), Z_t)dt + g(\bar{r}(t), Z_t)dW_t. \tag{10}$$

Applying Itô formula to $|Q_{t\wedge\upsilon_n}|^2$ yields

$$\begin{aligned}|Q_{t\wedge\upsilon_n}|^2 &= |Q_0|^2 + 2\int_0^{t\wedge\upsilon_n}\left\langle -\frac{\partial Q_s}{\partial a} - \mu(s,a)Z_s, Q_s\right\rangle ds + 2\int_0^{t\wedge\upsilon_n}(f(\bar{r}(s), Z_s), Q_s)ds \\ &\quad + 2\int_0^{t\wedge\upsilon_n}(Q_s, g(\bar{r}(s), Z_s)dW_s) + \int_0^{t\wedge\upsilon_n}\|g(\bar{r}(s), Z_s)\|_2^2 ds \\ &\le |Q_0|^2 - 2\int_0^{t\wedge\upsilon_n}\left\langle \frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds - 2\mu_0\int_0^{t\wedge\upsilon_n}(Z_s, Q_s)ds \\ &\quad + 2\int_0^{t\wedge\upsilon_n}(f(\bar{r}(s), Z_s), Q_s)ds + \int_0^{t\wedge\upsilon_n}\|g(\bar{r}(s), Z_s)\|_2^2 ds + 2\int_0^{t\wedge\upsilon_n}(Q_s, g(\bar{r}(s), Z_s)dW_s).\end{aligned}$$

Therefore, by (7), we get that

$$\begin{aligned}|Q_{t\wedge\upsilon_n}|^2 &\le |Q_0|^2 + A^2\bar{\beta}^2\int_0^{t\wedge\upsilon_n}|Q_s|^2 ds + 2\mu_0\int_0^{t\wedge\upsilon_n}|Q_s||Z_s|ds \\ &\quad + 2\int_0^{t\wedge\upsilon_n}|Q_s||f(\bar{r}(s), Z_s)|ds + \int_0^{t\wedge\upsilon_n}\|g(\bar{r}(s), Z_s)\|_2^2 ds + 2\int_0^{t\wedge\upsilon_n}(Q_s, g(\bar{r}(s), Z_s)dW_s).\end{aligned}$$

Applying inequality $2ab \le a^2 + b^2$, it follows that for any $t \in [0, T]$

$$\begin{aligned}E\sup_{0\le s\le t}|Q_{s\wedge\upsilon_n}|^2 &\le E|Q_0|^2 + (A^2\bar{\beta}^2 + 2\mu_0 + 1)\int_0^{t\wedge\upsilon_n}E\sup_{0\le s\le t}|Q_s|^2 ds \\ &\quad + \int_0^{t\wedge\upsilon_n}E|f(\bar{r}(s), Z_s)|^2 ds + \int_0^{t\wedge\upsilon_n}E\|g(\bar{r}(s), Z_s)\|_2^2 ds \\ &\quad + 2E\sup_{0\le s\le t}\int_0^{s\wedge\upsilon_n}(Q_u, g(\bar{r}(u), Z_u)dW_u).\end{aligned}$$

Using condition (ii) yields

$$\begin{aligned}E\sup_{0\le s\le t}|Q_{s\wedge\upsilon_n}|^2 &\le E|Q_0|^2 + (A^2\bar{\beta}^2 + 2\mu_0 + 1)\int_0^{t\wedge\upsilon_n}E\sup_{0\le s\le t}|Q_s|^2 ds \\ &\quad + 2K_i^2\int_0^{t\wedge\upsilon_n}E\|Z_s\|_C^2 ds + 2E\sup_{0\le s\le t}\int_0^{s\wedge\upsilon_n}(Q_u, g(\bar{r}(u), Z_u)dW_u).\end{aligned} \tag{11}$$

Applying the Burkholder–Davis–Gundy inequality again, we have

$$\begin{aligned}E\left[\sup_{0\le s\le t}\int_0^s(Q_u, g(\bar{r}(u), Z_u)dW_u)\right] &\le 3E\left[\sup_{0\le s\le t}|Q_s|\left(\int_0^t\|g(\bar{r}(s), Z_s)\|_2^2 ds\right)^{1/2}\right] \\ &\le \frac{1}{4}E\left[\sup_{0\le s\le t}|Q_s|^2\right] + K_1\int_0^t\|g(\bar{r}(s), Z_s)\|_2^2 ds \\ &\le \frac{1}{4}E\left[\sup_{0\le s\le t}|Q_s|^2\right] + K_1\cdot K_i^2\int_0^t E\|Z_s\|_C^2 ds,\end{aligned} \tag{12}$$

for some positive constant $K_1 > 0$. Thus, it follows from (11) and (12)

$$E\sup_{0\le s\le t}|Q_{s\wedge\upsilon_n}|^2 \le 2(A^2\bar{\beta}^2 + 2\mu_0 + 1 + 2K_i^2 + 2K_1K_i^2)\int_0^t E\sup_{0\le u\le s}|Q_{u\wedge\upsilon_n}|^2 ds + 2E|Q_0|^2, \quad \forall t \in [0, T].$$

Applying Gronwall's lemma, one can get

$$E\sup_{0\le s\le t}|Q_{s\wedge\upsilon_n}|^2 \le 2e^{2T(A^2\bar{\beta}^2+2\mu_0+1+2K_i^2+2K_1K_i^2)}E|Q_0|^2 = C_2, \quad \forall t \in [0, T].$$

The proof is finished. □

Lemma 4. *For any $t \in [0, T]$*

$$E\int_0^{t\wedge\upsilon_n} |f(\bar{r}(s), Q_s) - f(r(s), Q_s)|^2 \mathrm{d}s \le C_3 h + o(h),$$
$$E\int_0^{t\wedge\upsilon_n} \|g(\bar{r}(s), Q_s) - g(r(s), Q_s)\|_2^2 \mathrm{d}s \le C_4 h + o(h).$$

The proof is similar to that in [10].

Lemma 5. *Under the conditions* (ii)–(iii), *there exist constants $k \ge 2$ and $C_5 > 0$ such that*

$$E\sup_{t\in[0,T]} |Q_t|^k \le C_5.$$

The proof is similar to that of Lemma 2.

Lemma 6. *Under the conditions* (ii)–(iii) *and $E|\frac{\partial Q_s}{\partial a}|^2 < \infty$, then*

$$\int_0^{t\wedge\upsilon_n} E|Q_s - Z_s|^2 \mathrm{d}s \le C_6 h. \tag{13}$$

Proof. For $\forall t \in [0, T]$, there exists an integer k such that $t \in [kh, (k+1)h)$. We have

$$\begin{aligned} Q_t - Z_t = Q_t - Q_t^k &= -\int_{kh}^t \frac{\partial Q_s}{\partial a}\mathrm{d}s - \int_{kh}^t \mu(s,a)Z_s \mathrm{d}s + \int_{kh}^t f(\bar{r}(s), Z_s)\mathrm{d}s + \int_{kh}^t g(\bar{r}(s), Z_s)\mathrm{d}W_s \\ &= -\frac{\partial Q_t}{\partial a}(t - t_k) - \mu(t,a)Z_t(t-t_k) + f(\bar{r}(t), Z_t)(t-t_k) + g(\bar{r}(t), Z_t)(W_t - W_{t_k}). \end{aligned}$$

Applying $(a+b+c+d)^2 \le 4a^2 + 4b^2 + 4c^2 + 4d^2$ and condition (ii), we have

$$\begin{aligned} |Q_t - Z_t|^2 &\le 4\left(\frac{\partial Q_t}{\partial a}\right)^2 h^2 + 4\mu^2(t,a)|Z_t|^2 h^2 + 4|f(\bar{r}(t), Z_t)|^2 h^2 + 4|g(\bar{r}(t), Z_t)|^2 (W_t - W_{t_k})^2 \\ &\le 4\left(\frac{\partial Q_t}{\partial a}\right)^2 h^2 + 4\bar{\alpha}^2 |Q_t|^2 h^2 + 4K^2|Q_t|^2 h^2 + 4K^2 |Q_t|^2 (W_t - W_{t_k})^2. \end{aligned}$$

By Lemma 5 and Brown property, it follows

$$\begin{aligned} \int_0^{t\wedge\upsilon_n} E|Q_s - Z_s|^2 \mathrm{d}s &\le 4TE\left(\frac{\partial Q_t}{\partial a}\right)^2 h^2 + 4T\bar{\alpha}^2 E|Q_t|^2 h^2 + 4TK^2 E|Q_t|^2 h^2 + 4TK^2 E|Q_t|^2 E(W_t - W_{t_k})^2 \\ &\le 4TE\left(\frac{\partial Q_t}{\partial a}\right)^2 h + 4T\bar{\alpha}^2 C_5 h + 4TK^2 C_5 h + 4TK^2 C_5 h, \end{aligned}$$

where $K = \max_{i\in S} K_i$. The required result is obtained with

$$C_6 = 4T\left[E\left(\frac{\partial Q_t}{\partial a}\right)^2 + \bar{\alpha}^2 C_5 + 2K^2 C_5\right]. \quad \Box$$

4. Main results

Now we are in position to establish the following main results.

Theorem 7. *Under the conditions* (i)–(iii), *then*

$$E\sup_{0\le t\le T} |P_{t\wedge\upsilon_n} - Q_{t\wedge\upsilon_n}|^2 \le C_7 h + o(h). \tag{14}$$

Proof. Combining (3) with (6) has

$$\begin{aligned}P_t - Q_t &= -\int_0^t \frac{\partial(P_s - Q_s)}{\partial a}\mathrm{d}s - \int_0^t \mu(s,a)(P_s - Z_s)\mathrm{d}s + \int_0^t (f(r(s),P_s) - f(\bar r(s),Z_s))\mathrm{d}s \\ &\quad + \int_0^t (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s.\end{aligned}$$

Therefore using Itô formula, along with the Cauchy–Schwarz inequality and (7) yields,

$$\begin{aligned}|P_t - Q_t|^2 &= -2\int_0^t \left\langle P_s - Q_s, \frac{\partial(P_s - Q_s)}{\partial a}\right\rangle \mathrm{d}s - 2\int_0^t (P_s - Q_s, \mu(s,a)(P_s - Z_s))\mathrm{d}s \\ &\quad + 2\int_0^t (P_s - Q_s, f(r(s),P_s) - f(\bar r(s),Z_s))\mathrm{d}s + \int_0^t \|g(r(s),P_s) - g(\bar r(s),Z_s)\|_2^2\mathrm{d}s \\ &\quad + 2\int_0^t (P_s - Q_s, (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s) \\ &\le (A^2\bar\beta^2 + 3\bar\alpha + 2K^2 + 1)\int_0^t |P_s - Q_s|^2\mathrm{d}s + \bar\alpha\int_0^t |Q_s - Z_s|^2\mathrm{d}s \\ &\quad + 2K\int_0^t |P_s - Q_s|\|P_s - Z_s\|_C\mathrm{d}s + \int_0^t [f(r(s),Q_s) - f(\bar r(s),Z_s)]^2\mathrm{d}s \\ &\quad + 2\int_0^t \|g(r(s),Q_s) - g(\bar r(s),Z_s)\|_2^2\mathrm{d}s + 2\int_0^t (P_s - Q_s, (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s).\end{aligned}$$

Hence, by Lemmas 4 and 6, for any $t \in [0,T]$,

$$\begin{aligned}E\sup_{s\in[0,t]}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|^2 &\le (A^2\bar\beta^2 + 3\bar\alpha + 2K^2 + 1)\int_0^t E\sup_{u\in[0,s]}|P_{u\wedge\upsilon_n} - Q_{u\wedge\upsilon_n}|^2\mathrm{d}s \\ &\quad + 2\bar\alpha E\int_0^{t\wedge\upsilon_n} |Q_s - Z_s|^2\mathrm{d}s + \int_0^{t\wedge\upsilon_n} [f(r(s),Q_s) - f(\bar r(s),Z_s)]^2\mathrm{d}s + \int_0^{t\wedge\upsilon_n} \|g(r(s),Q_s) - g(\bar r(s),Z_s)\|_2^2\mathrm{d}s \\ &\quad + 2E\sup_{s\in[0,t]}\int_0^{s\wedge\upsilon_n} (P_s - Q_s, (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s) \\ &\le (A^2\bar\beta^2 + 3\bar\alpha + 2K^2 + 1)\int_0^t E\sup_{u\in[0,s]}|P_{u\wedge\upsilon_n} - Q_{u\wedge\upsilon_n}|^2\mathrm{d}s + 2\bar\alpha C_6 h \\ &\quad + (C_3 + C_4)h + o(h) + 2E\sup_{s\in[0,t]}\int_0^{s\wedge\upsilon_n} (P_s - Q_s, (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s). \qquad (15)\end{aligned}$$

By the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}&E\sup_{s\in[0,t]}\int_0^{s\wedge\upsilon_n} (P_s - Q_s, (g(r(s),P_s) - g(\bar r(s),Z_s))\mathrm{d}W_s) \\ &\le \mu_1 E\left[\sup_{0\le s\le t}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|\left(\int_0^{t\wedge\upsilon_n} \|g(r(s),P_s) - g(\bar r(s),Z_s)\|_2^2\mathrm{d}s\right)^{1/2}\right] \\ &\le \frac{1}{4}E\left[\sup_{0\le s\le t}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|^2 + \mu_2\int_0^{t\wedge\upsilon_n} E\|P_s - Q_s\|_C^2\mathrm{d}s + \mu_2 C_4 h + o(h)\right], \qquad (16)\end{aligned}$$

where μ_1, μ_2 are two positive constants. Therefore inserting (16) into (15) has

$$\begin{aligned}E\sup_{s\in[0,t]}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|^2 &\le (A^2\bar\beta^2 + 3\bar\alpha + 2K^2 + 1 + 2\mu_2)\int_0^t E\sup_{u\in[0,s]}|P_{u\wedge\upsilon_n} - Q_{u\wedge\upsilon_n}|^2\mathrm{d}s \\ &\quad + (2\bar\alpha C_6 + C_3 + C_4 + 2\mu_2 C_4)h + o(h) + \frac{1}{2}E\sup_{s\in[0,t]}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|^2.\end{aligned}$$

Applying Gronwall's inequality, we obtain a bound of the form

$$E\sup_{s\in[0,t]}|P_{s\wedge\upsilon_n} - Q_{s\wedge\upsilon_n}|^2 \le 2(2\bar\alpha C_6 + C_3 + C_4 + 2\mu_2 C_4)\mathrm{e}^{2T(A^2\bar\beta^2 + 3\bar\alpha + 2K^2 + 1 + 2\mu_2)}h + o(h).$$

The result then follows with

$$C_7 = 2(2\bar{\alpha}C_6 + C_3 + C_4 + 2\mu_2 C_4)e^{2T(A^2\bar{\beta}^2+3\bar{\alpha}+2K^2+1+2\mu_2)}. \qquad \Box$$

Theorem 8. *Under the conditions* (i)–(iii), *then*

$$E \sup_{0\le t\le T} |P_t - Q_t|^2 \le C_8 h + o(h).$$

Proof. Let $z_t = P_t - Q_t$. It is easy to see that

$$\begin{aligned}
E\left[\sup_{0\le t\le T}|z_t|^2\right] &= E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n>T \text{ and } \sigma_n>T\}}\right] + E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right] \\
&= E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\upsilon_n>T\}}\right] + E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right] \\
&\le E\left[\sup_{0\le t\le T}|z_{t\wedge\upsilon_n}|^2\right] + E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right].
\end{aligned} \tag{17}$$

By the Young inequality $xy \le \frac{x^p}{p} + \frac{x^q}{q}$ for any $a, b, p, q, \delta > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ab = a\delta^{1/p}\frac{b}{\delta^{1/p}} \le \frac{a^p\delta}{p} + \frac{b^q}{q\delta^{q/p}}.$$

Thus let $p = 2$, $\delta = h$, we have

$$E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right] \le \frac{h}{2}E\left[\sup_{0\le t\le T}|z_t|^4\right] + \frac{1}{2h}P\{\tau_n \le T \text{ or } \sigma_n \le T\}.$$

Note

$$E\left[\sup_{0\le t\le T}|z_t|^4\right] \le 8\left(E\left[\sup_{0\le t\le T}|P_t|^4\right] + E\left[\sup_{0\le t\le T}|Q_t|^4\right]\right) \le 8(C_1 + C_5).$$

$$P\{\tau_n \le T\} = E\left[\mathbf{1}_{\{\tau_n\le T\}}\frac{|P_{\tau_n}|^4}{n^4}\right] \le \frac{C_1 A}{n^4}.$$

$$P\{\sigma_n \le T\} = E\left[\mathbf{1}_{\{\sigma_n\le T\}}\frac{|Q_{\sigma_n}|^4}{n^4}\right] \le \frac{C_5 A}{n^4}.$$

Then

$$E\left[\sup_{0\le t\le T}|z_t|^2\mathbf{1}_{\{\tau_n\le T \text{ or } \sigma_n\le T\}}\right] \le 4h(C_1 + C_5) + \frac{1}{2hn^4}(C_1 + C_5).$$

By Theorem 7, (17) becomes

$$E\left[\sup_{0\le t\le T}|z_t|^2\right] \le C_7 h + o(h) + 4h(C_1 + C_5) + \frac{1}{2hn^4}(C_1 + C_5).$$

Let $n \ge (2h^2)^{-\frac{1}{4}}$, then

$$E\left[\sup_{0\le t\le T}|z_t|^2\right] \le C_7 h + o(h) + 4h(C_1 + C_5) + h(C_1 + C_5) = C_8 h + o(h).$$

The proof is finished with $C_8 = 5(C_1 + C_5) + C_7$. $\Box$

Theorem 9. *Under the conditions* (i)–(iii), *the numerical approximate solution* (6) *will converge to the exact solution to Eq.* (2) *in the sense*

$$\lim_{h\to 0} E\left[\sup_{0\le t\le T}|P_t - Q_t|^2\right] = 0.$$

The proof is easily deduced from Theorem 8.

Note. Let $\varepsilon \in (0, 1)$, $t \in [0, T]$, then it is seen from Theorem 8 that the total running time of the discrete numerical scheme is bounded by $O(1/\varepsilon^2)$ arithmetic operations.

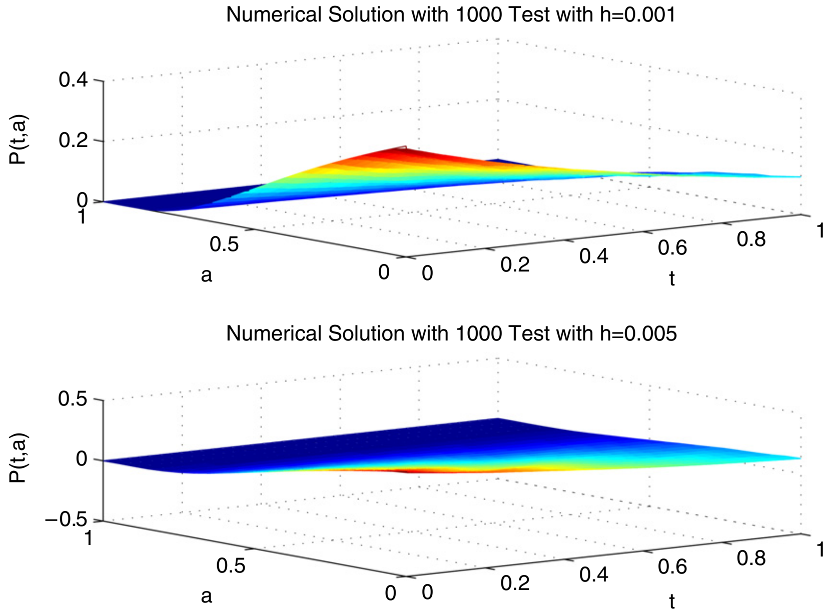

Fig. 1. Numerical simulations of the stochastic population equation.

5. An example

In this section we shall discuss an example to illustrate our theory.

Example. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Of course $w(t)$ and $r(t)$ are assumed to be independent. Let us consider a stochastic age-dependent population equation with Markovian switching of the form

$$\begin{cases} \mathrm{d}_t P = \left[-\dfrac{\partial P}{\partial a} - \dfrac{1}{(1-a)^2}P + f(r(t), P)\right]\mathrm{d}t + g(r(t), P)\mathrm{d}w_t, & (t, a) \in Q, \\ P(0, a) = \exp\left(-\dfrac{1}{1-a}\right), \; r(0) = i_0 = 1, & a \in [0, 1], \\ P(t, 0) = \displaystyle\int_0^1 \dfrac{1}{(1-a)^2} P(t, a)\mathrm{d}a, & t \in [0, T]. \end{cases} \tag{18}$$

Here $Q = (0, T) \times (0, 1)$, $\mu(t, a) = \beta(t, a) = \frac{1}{(1-a)^2}$, $f(1, P) = -P$, $f(2, P) = \arcsin(P)$, $g(1, P) = \sin(P)$ and $g(2, P) = \arctan(P)$, $P_0(a) = \exp(-\frac{1}{1-a})$.

It is easy to verify that the conditions (i)–(iii) are satisfied. Then, the approximate solution will converge to the true solution of (18) for any $(t, a) \in (0, T) \times (0, 1)$ in the sense of Theorem 9.

Obviously, $P(t, a)$ in (18) cannot be solved explicitly. It is necessary to know the numerical approximation $Q(t, a)$ of $P(t, a)$. Take $T = 1$, $\Delta a = 0.05$. Fig. 1 presents the numerical simulations of the stochastic age-dependent population equations with Markovian switching (18) for 1000 experiments (up: $\Delta t = h = 0.001$, down: $\Delta t = h = 0.005$), where $P(t, a) = EQ(t, a) = \frac{1}{1000}\sum_{k=1}^{1000} Q_k(t, a)$. The running times are $1.2210e-006$ and $9.6540e-007$, respectively. This clearly reveals the population dynamics tendency.

6. Conclusion

Recently, stochastic modelling with Markovian switching have received a great deal of attention. Most stochastic modelling with Markovian switching are nonlinear and cannot have explicit solutions, so the construction of efficient computational methods is of great importance. In this paper, a class of stochastic age-dependent population equations with Markovian switching is considered, the main purpose of this paper is to investigate the convergence of numerical approximation of stochastic age-dependent population equations with Markovian switching under the given conditions. An example is provided to illustrate our theory effectiveness.

Acknowledgements

The authors would like to thank two anonymous referees for their helpful comments and suggestions which greatly improved this paper. The work is partially supported by the China Petroleum University (Y060804).

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